NONLINEAR PHYSICS AND MECHANICS

Excitation of Large-Amplitude Localized Nonlinear Waves by the Interaction of Kinks of the Sine-Gordon Equation with Attracting Impurity

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The generation and evolution of localized waves on an impurity in the scattering of a kink of the sine-Gordon equation are studied. It is shown that the problem can be considered as excitation of oscillations of a harmonic oscillator by a short external impulse. The external impulse is modeled by the scattering of a kink on an impurity. The influence of the modes of motion of a kink on the excitation energy of localized waves is numerically and analytically studied. The method of collective coordinate for the analytical study is used. The value of this energy is determined by the ratio of the impurity parameters and the initial kink velocity. It is shown that the dependence of the energy (and amplitude) of the generated localized waves

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on the initial kink velocity has only one maximum. This behavior is observed for the cases of point and extended impurities. Analytical expression for the amplitude of the localized wave in the case of point impurity is obtained. This allows controlling the excitation energy of localized waves using the initial kink velocity and impurity parameters. The study of the evolution of localized impurities under the action of an external force and damping has shown a good agreement with the nondissipative case. It is shown that small values of the external force have no significant effect on the oscillations of localized waves. An analytical expression for the logarithmic decrement of damping is obtained. This study may help to control the parameters of the excited waves in real physical systems.

1. Introduction

Soliton solutions of nonlinear differential equations attract increased attention of researchers because of an increasing use in physical applications [1–18]. For example, solitons of the sine-Gordon equation(the SGE) are frequently used. Only in solid state physics they describe domain boundaries in magnets, dislocations in crystals, fluxons in Josephson junctions. The sine-Gordon equation appears in the modeling of wave processes in rocks, molecular biology, models of field theory and particle physics [2, 3]. Accounting for the influence of perturbations on the SGE solutions leads to a change in the structure of solitons and to the excitation of the internal degrees of freedom of solitons. It is important to study the SGE solutions considering the perturbations arising in real physical applications. The effect of small perturbations on the SGE solutions can be studied using a well-developed perturbation theory for solitons [5, 19, 20]. The effect of perturbations can generally be investigated only with the help of numerical methods [20–23]. For example, the influence of a coordinate- and time-dependent external force has been studied [24–26].

Numerous works have studied spatially dependent perturbations in the sine-Gordon model [5, 6, 18–43]. Many papers study the effect of spatial modulation of the periodic potential (or the presence of impurities in the system) on the dynamics of the SGE solitons [1, 2, 5, 6, 19, 28]. The SGE model with impurities describes, for example, a multilayer ferromagnet [40–43]. The importance of impurity modes in the kink dynamics was shown in [5, 28, 36]. The structure and properties of localized nonlinear waves excited on an impurity were analyzed numerically in [21, 23, 32]. The case of several point impurities that are of interest in some physical applications was considered in [44] and even the case of spatially modulated harmonic potential was considered [31]. The possibility of exciting localized waves as a result of the kink-impurity interaction has been also investigated in some cases [5, 6]. At first, a simple classical particle model was used for the theoretical description of the kink-impurity interaction when the excitation of an impurity mode was neglected, i.e., a localized oscillatory state on the impurity [5]. Accounting for the excitation of impurity modes in the kink-impurity interaction leads to an interesting effect, i.e., the reflection of a kink by an attracting impurity due to the resonance energy exchange between the translational mode of the kink and the impurity mode. The work [22] has shown that this effect can be observed under certain conditions even in the presence of damping that always occurs in real physical systems. The value of energy consumed for the excitation of localized waves by the kink will determine the value of a new contribution to the effective damping of a moving kink [45]. It may change depending on its initial velocity. It is known that, in this case, the frequency of excited localized nonlinear waves is practically

independent of the initial kink velocity. But this velocity must influence the amplitude of the excited wave. In this paper, we study the problem of the possibility of controlling the structure, dynamics, and energy of localized nonlinear waves excited by a moving kink on an impurity.

2. Basic equations and results

Consider a system determined by the Lagrangian

$$L = \int\limits_{-\infty}^{+\infty} \left\{ \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 - [1 - \varepsilon\delta(x)](1 - \cos u) + 4h\cos\frac{u}{2} \right\} dx. \tag{2.1}$$

The Rayleigh dissipation function has the form

$$R = \int\limits_{-\infty}^{+\infty} \frac{1}{2}\alpha\left(\frac{\partial u}{\partial t}\right)^2 dx. \tag{2.2}$$

The equation of motion for the scalar field $u(x,t)$ taking into account damping in the system takes the following form:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + [1 - \varepsilon\delta(x)]\sin u + 2h\sin\frac{u}{2} + \alpha\frac{\partial u}{\partial t} = 0. \tag{2.3}$$

where $\delta(x)$ is the Dirac delta function, ε is the constant, h is a value of the external force, α is a parameter of damping. Equation (2.3) is the modified sine-Gordon equation (MSGE).

In the case $\varepsilon = \alpha = h = 0$ Eq. (2.3) has a solution in the form of a topological soliton (or kink):

$$u(x,t) = 4\arctan\left(\exp\left[\Delta(v_0)^{-1}(x - v_0 t)\right]\right), \tag{2.4}$$

where $\Delta(v) = (1 - v^2)^{1/2}$, v_0 is a continuous parameter $(0 < v_0 < 1)$ determining the velocity of a kink. There is also a solution (2.3) in the form of a spatially localized soliton, namely, a quiescent breather:

$$u_{breather}(x,t) = 4\arctan\left(\frac{\sqrt{1-\omega^2}}{\omega} \cdot \frac{\sin\omega t}{\cosh\left[\sqrt{1-\omega^2}(x-x_0)\right]}\right), \tag{2.5}$$

where ω is the breather frequency and x_0 is the coordinate of its center.

In the research [5] the case of a "point impurity" was studied in detail $(\varepsilon\delta(x))$ and it was shown that for the corresponding sign of the constant ε it acts as an attracting potential on a kink, therefore, soliton can be localized. In the case of a spatially extended impurity the kink–impurity interaction was also studied in both undeformable and deformable kink models [41, 46, 47].

First, we consider the simplest case of point impurity. In this case, for ε, h, $\alpha \ll 1$, using the method of collective coordinate for the coordinate of the center $X = X(t)$ and the amplitude of localized wave $a = a(t)$ the following system of equations has been obtained (see, e.g., [22]):

$$\begin{cases} 8\ddot{X}(t) + U'(X) - \varepsilon a(t)F'(X) = -8\dot{X}(t)\alpha + 8h, \\ \quad\ddot{a}(t) + \Omega^2 a(t) - \frac{\varepsilon^2}{2}F(X) = -\dot{a}(t)\alpha, \end{cases} \tag{2.6}$$

where

$$F(X) = 2\frac{\sinh X(t)}{\cosh^2 X(t)}, \quad U(X) = 8 - \frac{2\varepsilon}{\cosh^2 X(t)},$$

and the frequency of "impurity mode" is determined in the form

$$\Omega^2 = 1 - \frac{\varepsilon^2}{4}. \tag{2.7}$$

The system of equations (2.6) is a system of second-order ordinary differential equations. It is much easier to study than the original equation (2.3). The first equation of (2.6) describes the dynamics of the kink taking into account the presence of the impurity and localized wave on it. The second equation of (2.6) describes the dynamics of a localized wave on an impurity taking into account the interaction of the kink.

2.1. Nondissipative case

Consider the problem of the generation of the impurity mode of maximum possible amplitude for the given parameters of the system. It is known that, when a kink is scattered by an impurity, part of its energy is spent on the excitation of a nonlinear wave localized on an impurity or a "breather" (if the kink passes through the impurity) [48, 49]. The value of this energy can vary depending on the initial kink velocity v_0. Since the frequency of the excited wave is practically independent of v_0 [50], v_0 should influence its amplitude. Consider the simplest case, i.e., the case of a point impurity in the framework of the model (2.6) for $h = \alpha = 0$. In Fig. 1 the result of the integration is given (2.6) with different modeling parameters, whence it is seen that the breather oscillation amplitude can vary significantly. Suppose that in this model the motion of a kink occurs according to the known law in the form

$$X_1(t) = v_0 t - 10. \tag{2.8}$$

If we consider the influence of impurities and neglect the only excitation of localized wave by analogy with solution (2.14), the law of motion of a kink can be represented as

$$X_2(t) = \operatorname{arcsinh}\left[\frac{\sqrt{v_0^2 + \varepsilon/2}}{v_0} \cdot \sinh\left(v_0 t - 10\right)\right]. \tag{2.9}$$
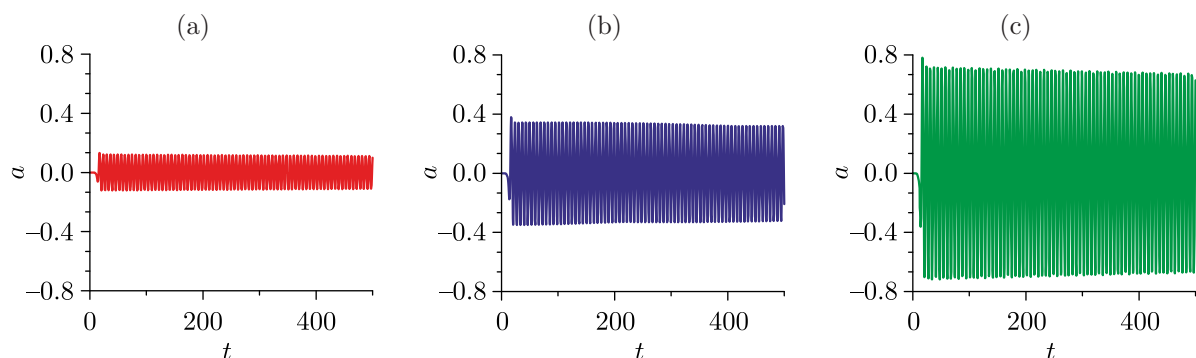

Fig. 1. Time dependence of $a(t)$ calculated by numerical integration (2.6) with initial conditions: $X(0) = -10$, $\dot{X}(0) = 0.7$, $a(0) = 0$, $\dot{a}(0) = 0$ and parameters (a) $\varepsilon = 0.3$, (b) $\varepsilon = 0.5$, (c) $\varepsilon = 0.7$.
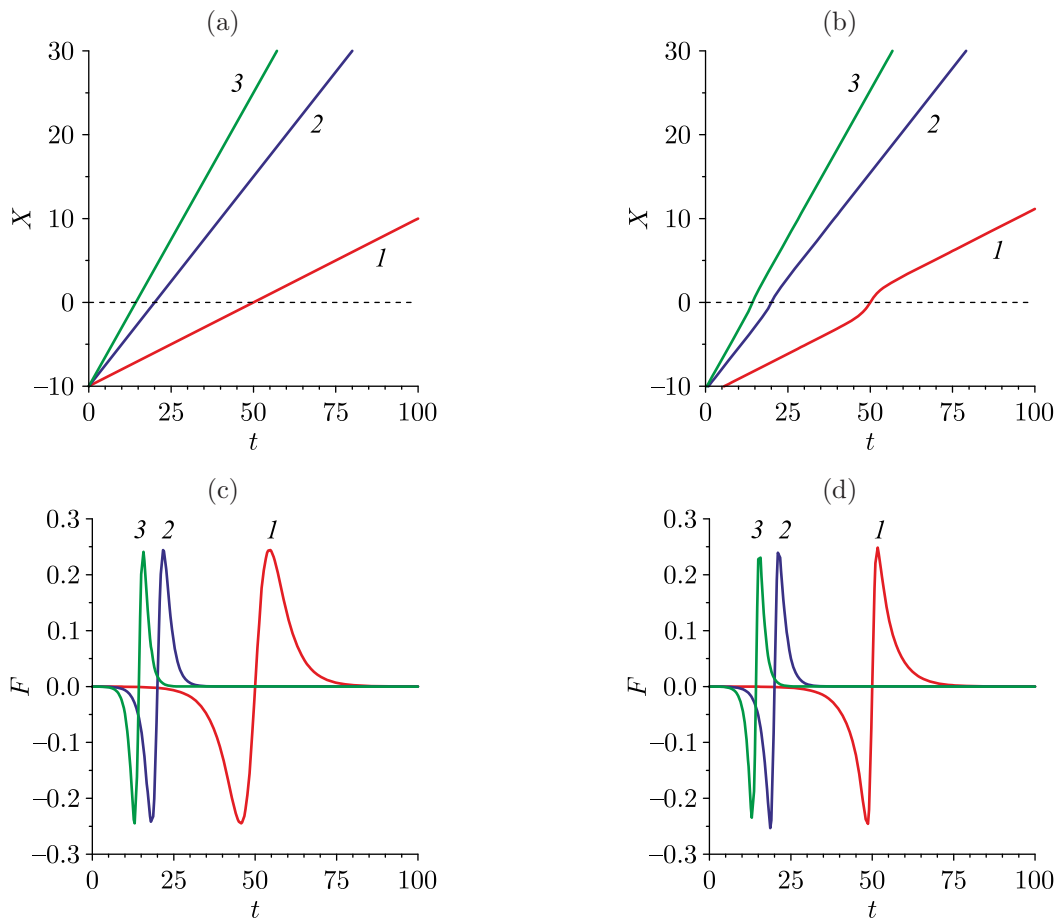
Fig. 2. $X(t) = X_1(t)$ (a) and $X(t) = X_2(t)$ (b) dependencies from expressions (2.8) and (2.9), respectively, and dependencies $F(t)$ at $X(t) = X_1(t)$ (c) and $X(t) = X_2(t)$ (d) constructed from the expression (2.11). The parameters are $\varepsilon = 0.7$: 1) $v_0 = 0.2$, 2) $v_0 = 0.5$, 3) $v_0 = 0.7$.

Figures 2a, 2b presents the graph of dependence $X(t)$ corresponding to (2.8)–(2.9) which shows that these dependences are slightly different for large values of v_0 (see curves 3 in Figs. 2a, 2b). However, in the region of small values of v_0 the dependence (2.9) shows a better qualitative correspondence with the results of numerical simulation (see curves 1 in Figs. 2a, 2b).

Since the collective coordinate $X(t)$ is described by the known law, the system (2.6) reduces to one equation for $a(t)$:

$$\ddot{a} + \Omega^2 a = F(t), \tag{2.10}$$

where

$$F(t) = \varepsilon^2 \frac{\sinh(X(t))}{\cosh^2(X(t))}. \tag{2.11}$$

Then this task can be considered as the excitation of a harmonic oscillator by an external force $F(t)$. Figures 2c, 2d present the dependence $F(t)$ for some particular cases. In this case, the initial conditions are assumed as follows:

$$X(0) = -10, \quad \dot{X}(0) = v_0, \quad a(0) = 0, \quad \dot{a}(0) = 0.$$

We use the Runge–Kutta method of numerical integration to study (2.8)–(2.11). The obtained time evolution of $a(t)$ is presented in Fig. 3, which shows that the oscillation amplitude $a(t)$ differs
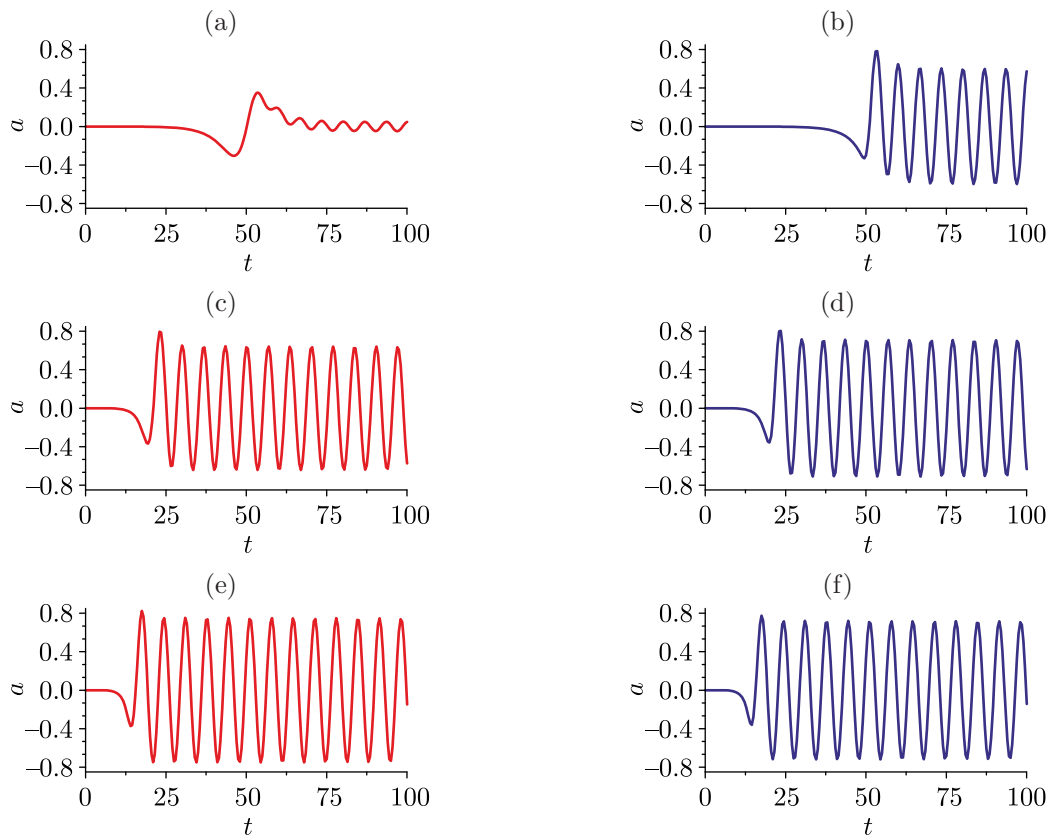
Fig. 3. Time dependence of $a(t)$ calculated by numerical integration (2.10), for the law of motion of a kink (2.8) (a), (c), (e) and (2.9) — (b), (d), (f) and parameters $\varepsilon = 0.7$: (a), (b) $v_0 = 0.2$, (c), (d) $v_0 = 0.5$, (e), (f) $v_0 = 0.7$.

significantly in the case of small v_0 (Figs. 3a, 3b). We also calculate the maximum breather oscillation amplitude $A_{\max}$ for cases (2.8) and (2.9) and compare it with the values obtained by numerical integration of the original system (2.6). Figure 4 shows that for $v_0 \to 1$ when the energy of the excited breather is considerably less than the total energy of a kink, both laws of motion of a kink (2.8) and (2.9) represent well the calculations of the original system (2.6). However, for small v_0 when the laws of motion differ significantly (see Figs. 2a, 2b, curve 1), the law of motion $X(t) = X_2(t)$ (2.9) gives considerably more accurate results. Therefore, it is used in further calculations.

It is possible to find an analytical expression for the breather oscillation amplitude in the case of a point defect using the law of conservation of energy. Then the localized oscillatory mode is given as in [6]:

$$u(x,t) = B(t) \exp(-\varepsilon|x|/2), \qquad (2.12)$$

where $B(t) = a_0 \cos(\Omega t + \theta_0)$, Ω is a frequency of the impurity mode (2.7), θ_0 being an initial phase. Then the energy stored by the breather can be calculated [5, 51]:

$$E_{im} = \frac{1}{2} \int\limits_{-\infty}^{+\infty} \left\{ \left(\frac{\partial u_{im}}{\partial t} \right)^2 + \left(\frac{\partial u_{im}}{\partial x} \right)^2 + [1 - \varepsilon \delta(x)] u_{im}^2 \right\} dx = \frac{\Omega^2 a_0^2}{\varepsilon}. \qquad (2.13)$$
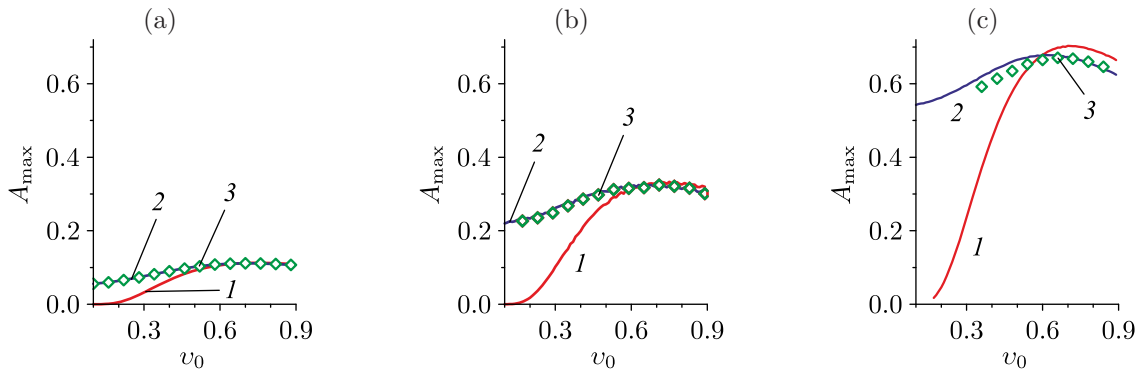
Fig. 4. Maximum breather oscillation amplitude $A_{\max}$ depending on the initial kink velocity v_0 at $\varepsilon = 0.3$ (a), $\varepsilon = 0.5$ (b) and $\varepsilon = 0.7$ (c), calculated by numerical integration: 1) Eqs. (2.10) for the law of motion of a kink (2.8), 2) Eqs. (2.10) for the law of motion of a kink (2.9), 3) systems of equations (2.6).

Suppose that the variation $X(t)$ is described by the solution [6]:

$$X(t) = \operatorname{arcsinh}\left(\gamma \sinh(v_0 t)\right), \tag{2.14}$$

where $\gamma = \sqrt{1 + \varepsilon/(2v_0^2)}$. Then the force acting on the breather (oscillator) is given by the expression (taking into account (2.11)):

$$f(t) = \frac{\varepsilon^2 A \sinh\left(v_0 t\right)}{1 + A^2 \sinh\left(v_0 t\right)}. \tag{2.15}$$

If the complex variable $\xi(t) = \dot{a} + i\Omega t$ is used, then the equation of oscillations (2.10) can be reduced to the form

$$\dot{\xi} - i\Omega\xi(t) = f(t), \tag{2.16}$$

which has the solution [51]:

$$\xi(t) = e^{i\Omega t} \int_{-\infty}^{t} f(\tau)e^{-i\Omega\tau}d\tau, \tag{2.17}$$

with initial conditions $a\left(-\infty\right) = \dot{a}\left(-\infty\right) = 0$, i.e., the oscillator (breather) is absent before the interaction. The total energy transferred from the particle (kink) to the oscillator (breather) can be found as in [51]:

$$E_{im} = 2\pi^2\varepsilon^2\sinh^2\left[\frac{\Omega}{2v_0}\arccos\left(\frac{2v_0^2 - \varepsilon}{2v_0^2 + \varepsilon}\right)\right]\cosh^{-2}\left(\frac{\Omega\pi}{2v_0}\right). \tag{2.18}$$

Then we obtain the following expression from (2.13) and (2.18) for the breather amplitude:

$$a_0^2 = \frac{2\pi^2\varepsilon^3}{\Omega^2}\sinh^2\left[\frac{\Omega}{2v_0}\arccos\left(\frac{2v_0^2 - \varepsilon}{2v_0^2 + \varepsilon}\right)\right]\cosh^{-2}\left(\frac{\Omega\pi}{2v_0}\right). \tag{2.19}$$

The expression (2.19) describes well the results obtained by numerical integration of the system (2.6) (Fig. 5). However, the dependences obtained are very far from the breather amplitude of the model (2.3). It was calculated by the numerical method described in [21]. Nevertheless,
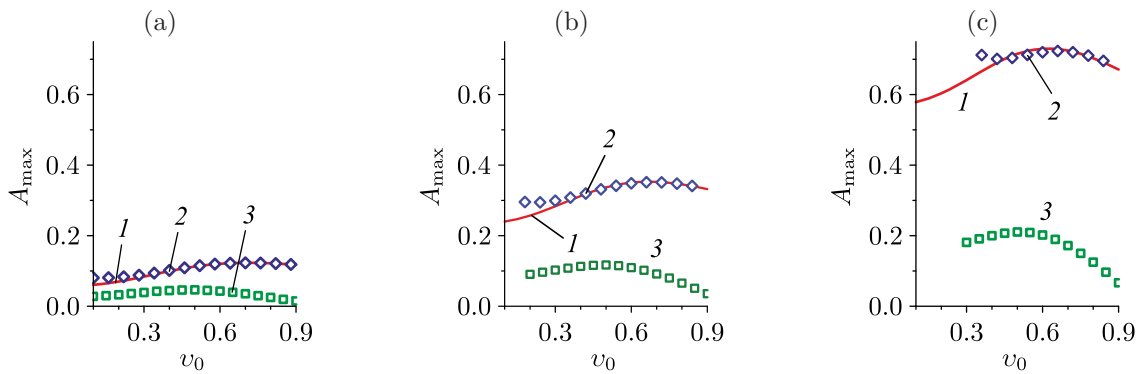
Fig. 5. Maximum breather oscillation amplitude $A_{\max}$ depending on the initial kink velocity v_0 at $\varepsilon = 0.3$ (a), $\varepsilon = 0.5$ (b) and $\varepsilon = 0.7$ (c), calculated: 1) by analytical expressions (2.19), 2) by numerical integration of the system of equations (2.6), 3) numerical simulation of Eq. (2.3).
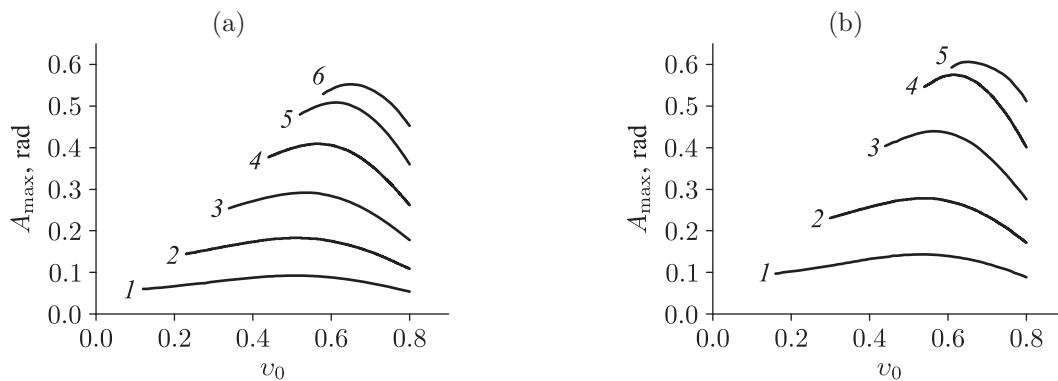

Fig. 6. Maximum breather amplitude $A_{\max}$ in the center of the extended impurity depending on the initial kink velocity v_0 at $W = 1$ (a) and $W = 1.5$ (b) in the framework of the model (2.20). Curves 1–6 correspond to the cases $\Delta K = 0.5, 0.75, 1, 1.25, 1.5, 1.75$, respectively.

the nature of the dependences is qualitatively the same: one maximum is present in all the curves in Fig. 5, which is presumably related to the ratio of the shape of the external force $F(t)$ and the shape of the potential $U(X)$. Moreover, the maximum point depends on the parameters of the impurity. Similar behavior is observed for the case of extended impurities of rectangular form with a width W and a depth ΔK, ($\Delta K = 0$ at $x < 0$, $x > W$):

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + [1 - \Delta K]\sin u + 2h\sin\frac{u}{2} + \alpha\frac{\partial u}{\partial t} = 0. \tag{2.20}$$

Figure 6 shows the maximum amplitude of the generated breather obtained numerically from Eq. (2.20). It is also shown that all curves have one maximum. The change in amplitude in this case is more significant than in the case of a point impurity.

2.2. Interaction of a kink with an impurity mode taking into account dissipation and external force

Consider the solution of the system of equations (2.6) in the presence of damping and external force. Since parameter h is absent in the second equation of the system (2.6) (due to the chosen approximation for the effective Lagrangian), the action of the external force does
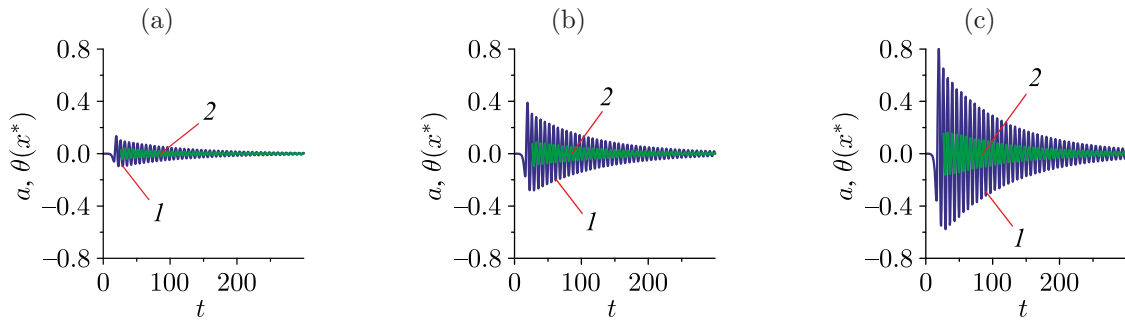
Fig. 7. Time evolution of the collective coordinate $a(t)$ (curve 1) calculated from (2.6) and $\theta(x^*)$ (curve 2) calculated from (2.3), for the case where the kink crosses by inertia the impurity with center at the point $x^* = 0$. The simulation parameters are $v_0 = 0.7$, $\alpha = 0.02$: (a) $\varepsilon = 0.3$, (b) $\varepsilon = 0.5$, (c) $\varepsilon = 0.7$.

not affect the oscillations of the impurity mode within this analytical model. Therefore, in this section, we restrict ourselves to the assumption that the kink moving by inertia passes through the impurity region.

Figure 7 presents the result of the integration of the analytical model (2.6) (curves 1) and the initial equation (2.3) (curves 2) for three different values of ε. It shows that the oscillation amplitude decreases under the effect of damping and the curves 1 and 2 are qualitatively the same. However, a quantitative comparison of their amplitude varies several times with increasing parameter ε.

Analysis of the analytical model (in the absence of dissipation) has shown that the excitation of an impurity mode as a result of the scattering of a kink on an impurity can be considered as the excitation of an effective oscillator by a short-term impulse $F(t)$. Figures 2c, 2d present an example of an approximate view of the dependence $F(t)$, where the graphs of this dependence are given for particular cases of the laws of motion of a kink $X(t)$. Thus, after the impurity mode is excited $F(t) \to 0$ further its oscillations occur under the influence of damping in the absence of external influence. Then the second equation of the system (2.6) can be represented (in our case) in the form

$$\ddot{a}(t) + \Omega^2 a(t) = -\dot{a}(t)\alpha, \qquad (2.21)$$

which is a well-known equation of oscillations of a harmonic oscillator with damping [52], its solution can be represented in the form

$$a(t) = B(t)\cos(\Omega t + \phi_0), \quad B(t) = B_0 \exp\left(-\frac{\alpha t}{2}\right), \qquad (2.22)$$

where ϕ_0 is an initial phase and $B(t)$ is an amplitude of the oscillations. Then the logarithmic decrement of damping λ of the dependence $a(t)$ taking into account the expression $\Omega^2 = 1 - \dfrac{\varepsilon^2}{4}$ can be found as follows:

$$\lambda = \frac{\alpha\pi}{\Omega} = \frac{\alpha\pi}{\sqrt{1 - \varepsilon^2/4}}. \qquad (2.23)$$

The formulas (2.22)–(2.23) describe sufficiently accurately the dependence $a(t)$ only from a particular moment of time. Figure 8 shows that the oscillation amplitude of the dependence $a(t)$ (solid curves) coincides quantitatively with the analytical expression for $B(t)$ (dashed curves), starting from the third oscillation period, since the effect of a nonzero value of the

Fig. 8. Curve 1 is a time evolution of collective coordinate of $a(t)$ from (2.6) for the case where the kink crosses by inertia the impurity at $v_0 = 0.7$, $\alpha = 0.02$: a) $\varepsilon = 0.5$, b) $\varepsilon = 0.7$. Curve 2 is a dependence of $B(t)$ calculated by the formula (2.22) at $\alpha = 0.02$: (a) $B_0 = 0.38169$, (b) $B_0 = 0.79306$.

impulse $F(t)$ is still manifested in the first two periods. Therefore, the initial part of the dependence should be excluded from the analysis when calculating the decrement of damping directly (for example, for the model (2.3)).

Next calculate the logarithmic decrement of damping λ of the oscillations of the impurity mode within the framework of the initial equation (2.3). Figure 9 presents the dependence $\lambda(\varepsilon)$, calculated by integration of Eq. (2.3) (solid curve 1) and the analytical dependence, constructed using the formula (2.23) (dashed curve 2). This figure shows that the curves obtained have a good quantitative agreement. The error of the curve is due to the fact that the amplitude of the excited impurity mode is extremely low at small values of ε (see Fig. 7). Therefore, the relative error in calculating the amplitude of the impurity mode is significant.

Thus, the effect of damping on the impurity mode is almost identical to the model (2.3) despite admitted approximation used in the construction of analytical models of (2.6). This suggests that the dissipation of the system, which takes into account the waste of energy on the excitation of the impurity mode, is one of the main damping channels (at least for the point impurities under consideration). However, it is possible that the radiation emitted by a localized wave can also play a significant role for large oscillation amplitudes of the impurity modes. Similar assumptions are discussed, for example, in [50, 53].
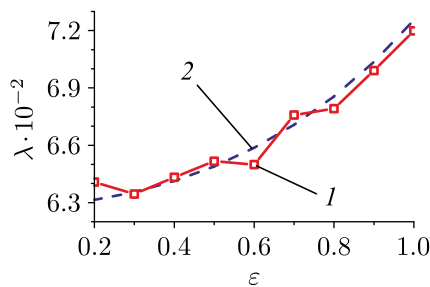

Fig. 9. Logarithmic decrement of damping λ depending on ε when $\alpha = 0.02$, $v_0 = 0.7$. Curve 1 is calculated in the integration of Eq. (2.3). Curve 2 is built according to the formula (2.23).

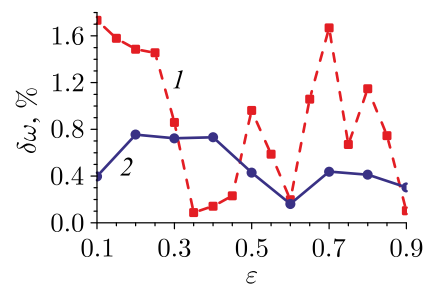Fig. 10. Dependence of the relative deviation of the oscillation frequency of the impurity mode $\delta\omega$ for a point impurity on the parameter ε at $\alpha = 0.02$. The deviation is calculated from the exact value, which is determined by the formula (2.7). The impurity mode is excited as the kink moves by inertia (curve 1) under the action of an external force (curve 2) $h = 0.0196039$ ($v_0 = 0.7$).

The significant difference in the amplitude of the excited impurity modes in both models (Fig. 7) is due to the approximations allowed in the derivation of (2.3).

The effect of an external force can have a certain effect on the oscillations of the impurity mode as it was shown in [54]. For example, the action of the field can lead to a change in the oscillation frequency, however, this change may be insignificant in the case of a small external force. Two graphs of the dependence of the relative error in the oscillation frequency of the impurity mode for the cases of motion of the kink under the action of an external force (curve 1) and by inertia (curve 2) are shown for comparison in Fig. 10. The occurrence of damping did not affect considerably the deviation of the frequency of the impurity mode $\delta\omega$ from the nondissipative case (in both cases $\delta\omega < 1\%$). Also, Fig. 10 shows that the relative deviation of the frequency of the impurity mode from the nondissipative case does not exceed 2% for the case of motion of the kink under the action of an external force $h = 0.0196039$. Thus, the presence of a small external force has no significant effect on the oscillations of the impurity mode. The approximation in the analytical model, when h is absent in the second equation of (2.6), is justified.

3. Conclusion

It was shown that the scattering of a kink on an impurity can be considered as an effective way to excite localized nonlinear waves. The dependence of the breather amplitude on the initial kink velocity when there is only one maximum was calculated for both point and extended impurities. This allows controlling the excitation energy of localized waves using the initial kink velocity and impurity parameters. This behavior remains qualitatively the same in the presence of a small external force and damping. The analytical expression for the decrement of damping was calculated. It is shown that the considered small values of the external force do not have a significant effect on the oscillations of localized waves. This opens up the possibility of using this method of generation of localized impurities in real physical systems. A small external force can be used to set the initial kink velocity.

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