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On the Possibility of the Observation of the Resonance Interaction between Kinks of the Sine-Gordon Equation and Localized Waves in Real Physical Systems

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The resonance dynamics of kinks of the sine-Gordon equation in a system with a single point impurity with allowance for the generation of localized waves in the presence of an external force and dissipation has been studied. Equations of motion for the coordinate of the kink center and the amplitude of the impurity mode have been found. The analysis of solutions of these equations and results of the numerical simulation of the modified sine-Gordon equation have showed that damping and the external force act against the appearance of the resonance reflection of the kink from the attractive impurity. However, the resonance energy exchange between solitons causing it still takes place.

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An increasing number of physical applications in different fields of physics have been described recently by the dynamics of solitons [1, 2]. For example, solitons of the sine-Gordon equation in condensed matter physics describe the domain boundaries in magnets, dislocations in crystals, fluxons in Josephson contacts and junctions, etc. [3]. The effect of perturbations often leads to a considerable variation of the soliton structure [1, 2]. The internal degrees of freedom of solitons can be excited, which can play the determining role in some physical effects. The effect of the spatial modulation (inhomogeneity) of the periodic potential (or the presence of an impurity in the system) on the dynamics of solitons of the sine-Gordon equation was studied in many works (see, e.g., [1–6]). The classical-particle model for the interaction between a kink and an impurity is applicable when the impurity does not allow the existence of an impurity mode, i.e., a localized oscillating state on the impurity [1, 2]. The importance of the impurity modes for the dynamics of the kink was shown in [4–7]. We note here the appearance of such an interesting effect as the reflection of the kink with inertial motion in a dissipationless medium by the "attracting" impurity owing to the resonance energy exchange between the translation mode of the kink and the impurity mode. However, experimental works on the observation of this effect have not yet been performed obviously because real physical systems are always characterized by dissipation, which can critically affect the behavior of a system. In this respect, it is necessary to study the influence of damping and the external force on the appearance of the resonance effects in the motion of kinks of the sine-Gordon equation in the model with the "attracting" impurity and to find the critical parameters of the real physical system suitable for the observation of such effects.

We consider a system determined by the Lagrangian

$$L = \int_{-\infty}^{\infty}\left\{\frac{u_t^2}{2} - \frac{u_x^2}{2} - K(x)(1-\cos u) + 4h\cos\frac{u}{2}\right\}dx, \quad (1)$$

where $K(x) = 1 - \varepsilon\delta(x)$ simulates a point impurity, $\delta(x)$ is the Dirac delta function, $0 < \varepsilon < 1$ is a constant, and h is the parameter determining the external force amplitude. To take into account damping in the system, we use the Rayleigh dissipation function:

$$R = \int_{-\infty}^{\infty}\frac{1}{2}\alpha u_t^2 dx. \quad (2)$$

We note that Eqs. (1) and (2) can describe, e.g., the dynamics of the domain boundaries in ferromagnets and weak ferromagnets [8]. The substitution of Eqs. (1) and (2) into Lagrange–Euler equations with allowance for dissipation leads to the equation of motion for the scalar field $u(x, t)$ in the form

$$u_{tt} - u_{xx} + [1 - \varepsilon\delta(x)]\sin u = 2h\sin\frac{u}{2} + \alpha u_t. \quad (3)$$

This equation is the modified sine-Gordon equation. In the absence of the impurity, external force, and damping, Eq. (3) transforms to the sine-Gordon

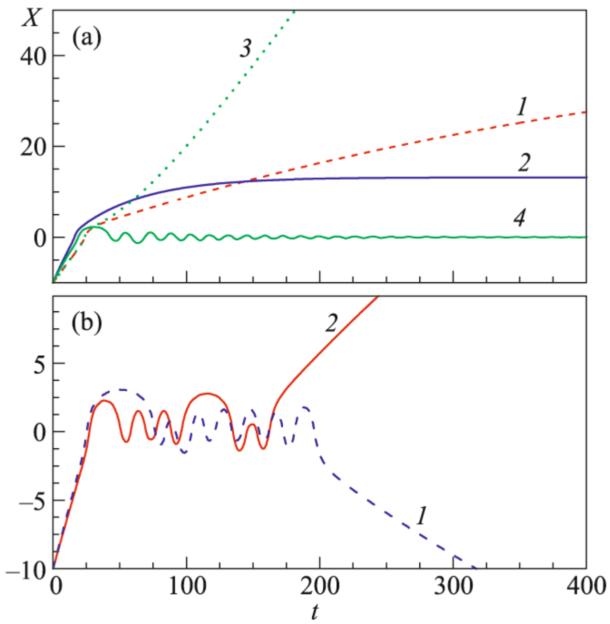

Fig. 1. (Color online) Time dependences of the coordinate of the kink center $X(t)$ obtained by the simulation of system (9). Simulation parameters are $\varepsilon = 0.7$, $X(0) = -10$, $a(0) = 0$, $\dot{a}(0) = 0$. (a) (*1*) $\alpha = 0.002$, $h = 0$, $\dot{X}(0) = 0.4$; (*2*) $\alpha = 0.02$, $h = 0$, $\dot{X}(0) = 0.7$; (*3*) $\alpha = 0.02$, $h = 0.008$, $\dot{X}(0) = 0.4$; (*4*) $\alpha = 0.02$, $h = 0$, $\dot{X}(0) = 0.6$; (b) (*1*) $\alpha = 0.002$, $h = 0$, $\dot{X}(0) = 0.377$; (*2*) $\alpha = 0.002$, $h = 0$, $\dot{X}(0) = 0.3486$.

equation and has the exact solution in the form of a topological soliton (or kink):

$$u_0(x, t) = 4\arctan(\exp\{\pm\gamma[x - X(t)]\}), \quad (4)$$

where $\gamma = (1 - v^2)^{-1/2}$, $0 < v < 1$ is the velocity of the kink, and $X(t) = vt + x_0$ is the coordinate of the kink center. At $v \ll 1$ and $\gamma \approx 1$, by solving the linearized sine-Gordon equation in the absence of the external force and damping, one can find the expression describing the structure of the impurity mode:

$$u_1(x, t) = a(t)\exp(-\varepsilon|x|/2), \quad (5)$$

where $a(t) = a_0\cos(\Omega t + \theta_0)$, $\Omega = \sqrt{1 - \varepsilon^2/4}$ is the frequency of the impurity mode, and θ_0 is the initial phase.

The case of the single point impurity disregarding the external force and decay was studied in detail in [1, 4]. It was shown that the impurity acts as a potential in the approximation of the "undeformable" kink. For the corresponding sign of the constant ε, it acts on the kink as an attracting potential. As a result, the soliton can be localized. For the "deformable kink" approximation, the possibility of the excitation of the impurity mode and its resonance interaction with the kink was taken into account. The interaction of the kink with the spatially extended impurity was also studied for both the undeformable and deformable kink models [5, 9, 10].

We consider the approximate analytical solution of Eq. (3) by the collective variable method [1, 2]. The coordinate of the kink center $x(t)$ and the impurity mode amplitude $a(t)$ are taken as collective coordinates. The ansatz is a sum of the kink given by Eq. (4) and the impurity mode specified by Eq. (5): $u_{\text{ansatz}} = u_0 + u_1$. For simplicity, we assume that $\gamma = 1$, and $\dot{X}(t)$, $a(t)$, and $\dot{a}(t)$ are sufficiently small (on the order of ε). In this approximation, $u_1 \ll u_0$. Then, we substitute the ansatz u_{ansatz} into Lagrangian (1), expanding $\cos u$ and $\cos u/2$ in a Taylor series in ε (in $a(t)$) up to the second order terms. Integration gives

$$L_{\text{eff}} \approx 4\dot{X}^2(t) - U(X) + \frac{1}{\varepsilon}[\dot{a}^2(t) - a^2(t)\Omega^2] + \varepsilon a(t)F(X) - 4h\int_{-\infty}^{\infty}\tanh[x - X(t)]dx, \quad (6)$$

where

$$U(X) = 8 - \frac{2\varepsilon}{\cosh^2 X(t)}, \quad F(X) = 2\frac{\sinh X(t)}{\cosh^2 X(t)}. \quad (7)$$

The expression for the Rayleigh function given by Eq. (2) is calculated analogously:

$$R_{\text{eff}} \approx \alpha\left[4\dot{X}^2(t) + \frac{\dot{a}^2(t)}{\varepsilon}\right]. \quad (8)$$

Using the Lagrange–Euler equations, taking into account Eqs. (6)–(8), one can obtain equations of motion for collective coordinates in the form

$$\begin{cases} 8\ddot{X}(t) + U'(X) - \varepsilon a(t)F'(X) = -8[\dot{X}(t)\alpha - h], \\ \ddot{a}(t) + \Omega^2 a(t) - \dfrac{\varepsilon^2}{2}F(X) = -\dot{a}(t)\alpha. \end{cases} \quad (9)$$

Comparison of the system of differential equations (9) with a similar system [1] obtained for the dissipationless case indicates that allowance for damping and the external force (within the considered approximation) leads to the addition of terms $-8[\dot{X}(t)\alpha - h]$ and $-\dot{a}(t)\alpha$ in the first and second equations, respectively. The effect of the external force on the impurity mode in this approximation is disregarded (as a term of second-order smallness) and the coefficient h in the second equation of the system is absent.

We analyze the behavior of the kink satisfying Eqs. (9). The typical simulation time was $t_{\text{end}} = 500$. Figure 1 shows the time evolution for several cases. The initial energy of the kink E^0_{kink} and the work of the external force E_{ex} are spent not only on the excitation of the impurity mode E_{im} but also on damping in the system E_α: $E^0_{\rm kink} + E_{\rm ex} = E_{\rm kink} + E_{\rm im} + E_\alpha$. Therefore, at the inertial motion ($h = 0$), the kink does not go to infinity at all and, after a certain time, stops at a certain point. This is seen in Fig. 1a (curve *1* for weak damping $\alpha = 0.002$ and curve *2* for stronger damping $\alpha = 0.02$).

If the external force $h > 0$ acts on the system and the dissipation losses E_α are compensated by the energy input $E_{\rm ex}$, the kink may go to infinity ($+\infty$) if its energy is sufficient for it to leave the attracting potential of the impurity (see Fig. 1a, curve *3*).

The results showed that the resonance interaction between the kink and impurity mode is characteristic of system (9), as well as of the dissipationless case. As a result, the kink can even leave the attractive potential of the impurity after repeatedly crossing it. However, these variants of evolution are observed only at weak damping and the inertial motion of the kink. For example, in Fig. 1b plotted at $\alpha = 0.002$, the kink is reflected in the opposite direction after 14 crossings of the impurity (see curve *1*) and in the initial direction after 11 crossings of the impurity (see curve *2*). At stronger damping (e.g., $\alpha = 0.02$), similar variants of evolution were not observed. In addition, the kink can be “trapped” by the attractive potential of the impurity. The amplitude of its translational oscillations decreases quite rapidly (see Fig. 1a, curve *4*). The impact of the external force h does not lead to the appearance of the resonance reflections of the kink, and the arising cases of the kink trapping differ weakly from the case of $h = 0$ (see Fig. 1a, curve *4*).

A more complete picture of the kink–impurity interactions was shown by the example of two cases: at weak damping $\alpha = 0.002$ and strong damping $\alpha = 0.02$ (the other simulation parameters are $h = 0$, $\varepsilon = 0.7$, $X(0) = -10$, $a(0) = 0$, and $\dot{a}(0) = 0$). The corresponding dependences of the calculated quantities on the initial velocity of the kink $v_0 = \dot{X}(0)$ are shown in Figs. 2–4 with the step $\delta v = 10^{-4}$ of the parameter v_0.

Figure 2 shows the number of crossings of the impurity $C_{\rm im}$ as a function of v_0. The single crossing of the impurity $C_{\rm im} = 1$ (Fig. 1a, curves *1* and *2*) takes place only if v_0 exceeds a certain threshold value. At $C_{\rm im} > 1$, an even $C_{\rm im}$ value corresponds to the resonance reflection in the opposite direction (Fig. 1b, curve *1*) and an odd $C_{\rm im}$ value in the initial direction (Fig. 1b, curve *2*). The case where the kink remains localized on the impurity is absent in Fig. 2 since it corresponds to $C_{\rm im}(v_0) \to \infty$. The study showed that the appearance of weak damping (Fig. 2a) in the system significantly reduces the number of the resonance windows and, consequently, the $C_{\rm im}$ value. At strong damping, the resonances disappear completely (Fig. 2b).

It should be noted that the dependences $v_{\rm end}(v_0)$ can be disregarded when studying the resonance

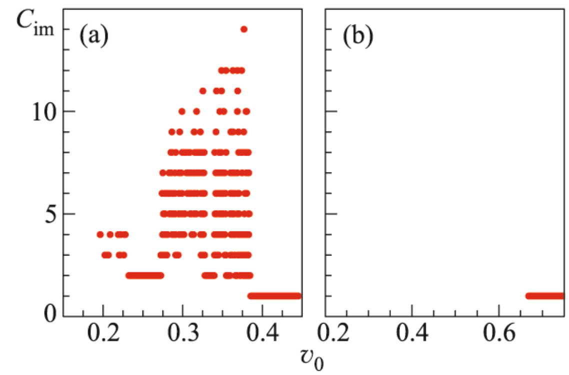

Fig. 2. (Color online) Number of crossings of the impurity by the kink $C_{\rm im}$ versus the initial velocity of the kink v_0 in model (9) with α = (a) 0.002 and (b) 0.02.

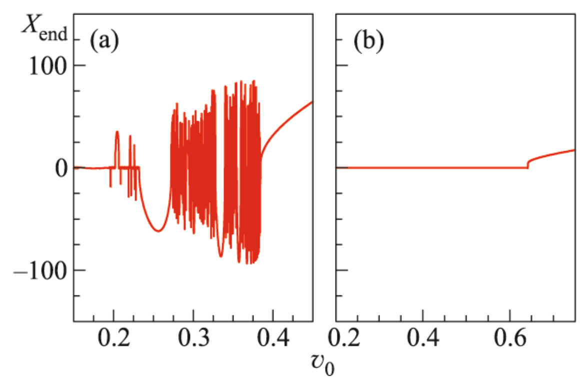

Fig. 3. (Color online) End position of the kink $X_{\rm end} = X(t_{\rm end})$ versus the initial velocity of the kink v_0 in model (9) with α = (a) 0.002 and (b) 0.02.

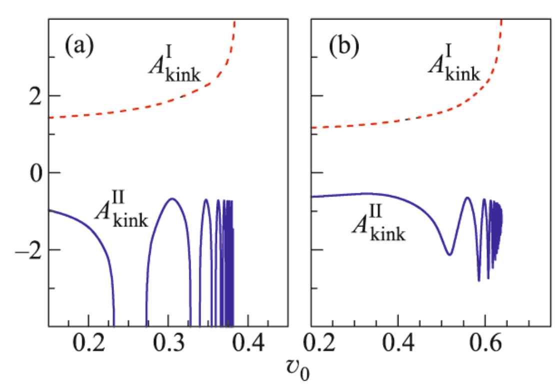

Fig. 4. (Color online) Maximum ($A^{\rm I}_{\rm kink}$) and minimum ($A^{\rm II}_{\rm kink}$) of the translational oscillations of the kink versus the initial velocity of the kink v_0 in model (9) with α = (a) 0.002 and (b) 0.02.

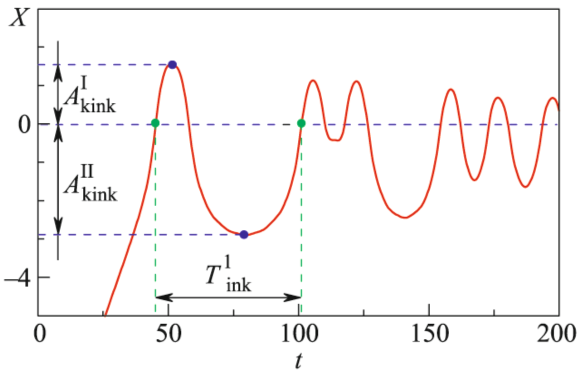

Fig. 5. (Color online) Time dependences of the coordinate of the kink center $X(t)$ obtained by the simulation of system (9). The determination of the maximum ($A^{\rm I}_{\rm kink}$) and minimum ($A^{\rm II}_{\rm kink}$) of the translational oscillations of the kink is shown; $T^1_{\rm kink}$ is the first period of the oscillations of the kink in the region of the impurity.

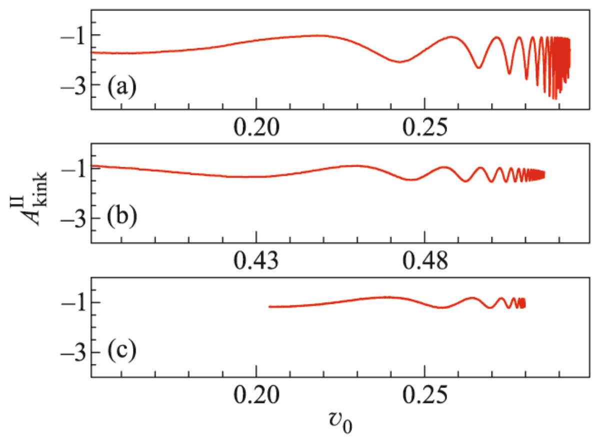

Fig. 6. (Color online) Minimum of the translational oscillations of the kink $A^{\rm II}_{\rm kink}$ versus the initial (stationary) velocity of the kink v_0 in model (3) in the case of (a, b) the inertial motion of the kink and (c) the motion under the impact of the external force h at α = (a) 0.002 and (b, c) 0.02.

reflection windows (see, e.g., [5, 6]) since the final velocity of the kink in the long-term simulation is either zero (in the case of inertial motion) or the stationary velocity $v_{\rm st}(h)$ (in the case of motion under the impact of the external force). For this reason, the use of the v_0 dependence of the final position of the kink $X_{\rm end} = X(t_{\rm end})$ provides more information (see Fig. 3). The case $X_{\rm end} < 0$ corresponds to the reflection of the kink from the impurity, the case $X_{\rm end} > 0$ corresponds to the passage of the kink through the impurity, and the case $X_{\rm end} = 0$ corresponds to the trapping of the kink on the impurity. The study of these dependences (analogous to $C_{\rm im}(v_0)$) showed that the increase in damping leads to the decrease in the number of resonance windows. At strong damping (Fig. 3b), the resonances disappear completely.

The amplitude of the dependence $X(t)$ during the first period of the kink oscillations $T^1_{\rm kink}$ provides more information. We consider the maximum ($A^{\rm I}_{\rm kink}$ at $C_{\rm im} \geq 2$) and minimum ($A^{\rm II}_{\rm kink}$ at $C_{\rm im} \geq 3$) of the dependence $X(t)$ in this interval (Fig. 5). Figure 4 shows the dependences $A^{\rm I}_{\rm kink}(v_0)$ and $A^{\rm II}_{\rm kink}(v_0)$. It can be seen that $A^{\rm I}_{\rm kink}$ increases monotonically with v_0, while $A^{\rm II}_{\rm kink}$ shows a periodic dependence on v_0. This is explained by the fact that $A^{\rm I}_{\rm kink}$ is calculated after the first crossing of the impurity, when the impurity mode is not excited, $a(t^*) = 0$, while $A^{\rm II}_{\rm kink}$ is calculated after the second crossing of the impurity by the kink. Already at the second crossing, the kink interacts with the excited impurity mode, $a(t^{**}) \neq 0$. It can be seen from the dependence $A^{\rm II}_{\rm kink}(v_0)$ that the wide windows ($C_{\rm im} = 2$) at weak damping (Fig. 4a) are transformed into small minima at strong damping (Fig. 4b). This indicates that the resonance interaction between the kink and impurity mode still takes place, but losses in the system leave no energy for the kink to leave the attractive potential of the impurity.

Further, we compare the results obtained using the analytical model given by Eq. (9) with the results of the direct numerical calculations using initial equation (3). We use the finite difference method for numerically solving Eq. (3). We chose the three-layer explicit scheme of the solution with the approximation of derivatives on the five-point template of the "cross" type (see, e.g., [9, 10]). We use two schemes of the numerical experiment (by analogy with the real physical experiment, which can be performed, e.g., in a real magnetic system [11]). The first scheme assumes that the kink at rest at the initial time instant is accelerated to a velocity close to the stationary one and then crosses the impurity. The second scheme makes it possible to study only the effect of damping. In this case, after the kink reaches a velocity close to the stationary one, the external force is switched off and the further motion of the kink through the region of the impurity localization occurs inertially. Since all the calculations below were performed according to the above schemes of the experiment, where the kink first is accelerated to the stationary velocity by the external force h, the velocity of the stationary motion of the kink characteristic of the given α and h values is taken as the initial velocity of the kink v_0: $v_0 = v_{\rm st}(h, \alpha) = \dot{X}(t) = h/\alpha$.

Figure 6 shows the dependences $A^{\rm II}_{\rm kink}(v_0)$ calculated with the step $\delta v = 3 \times 10^{-5}$ of the parameter v_0.

For the inertial motion of the kink (Figs. 6a and 6b), the situation analogous to that in Fig. 4b is observed. Even weak damping ($\alpha = 0.002$ in Fig. 6a) leads to the disappearance of the resonance reflection windows. This can be explained by the fact that the numerical calculation involves an additional damping channel in the form of the radiation of nonlocalized waves at the interaction between the kink and impurity as compared to Eq. (9) [9]. This leads to considerable losses of the kink energy even at slight damping. It was shown in [9] that this damping channel can be comparable with conventional damping. At stronger damping ($\alpha = 0.02$ in Fig. 6b), the amplitude of the translational oscillations of the kink decreases noticeably. A constant external force (see Fig. 6c) further weakens the effect of the resonance interaction, and the amplitude of the translational oscillations of the kink becomes lower than that in the case in Fig. 6b, since the external force hampers the motion of the kink in the opposite direction (at oscillations in the region of the impurity).

To conclude, we note that ferromagnets and weak ferromagnets (having weak damping) [8, 11] fully satisfy the above parameters for the possible observation of the resonance interaction of kinks of the sine-Gordon equation with localized waves. The creation of certain conditions, e.g., a three-layer structure with a central thin layer with the magnetic anisotropy lower (or of another type) than that in thick layers, upon crossing the thin layer by the domain boundary can lead to the generation of localized magnetic inhomogeneities of the breather type [12] and the resonance interaction between them. To find the resonance reflection and quasitunneling effects in real physical experiments, one can apply the method of measuring the amplitude of the translational oscillations of the kink localized in the region of the impurity and use physical systems with rather weak damping. Though a point impurity is impossible in a real system, the presence of an extended impurity in it is possible. This should also lead to the possibility of the observation of the sought effect.

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